Logarithmic lift of the $\widehat{su}(2)_{-1/2}$ model

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Abstract

This paper carries on the investigation of the non-unitary $\widehat{su}(2)_{-1/2}$ WZW model. An essential tool in our first work on this topic was a free-field representation, based on a $c = -2$ $\eta\xi$ ghost system, and a Lorentzian boson. It turns out that there are several 'versions' of the $\eta\xi$ system, allowing different $\widehat{su}(2)_{-1/2}$ theories. This is explored here in details. In more technical terms, we consider extensions (in the $c = -2$ language) from the small to the large algebra representation and, in a further step, to the full symplectic fermion theory. In each case, the results are expressed in terms of $\widehat{su}(2)_{-1/2}$ representations. At the first new layer (large algebra), continuous representations appear which are interpreted in terms of relaxed modules. At the second step (symplectic formulation), we recover a logarithmic theory with its characteristic signature, the occurrence of indecomposable representations. To determine whether any of these three versions of the $\widehat{su}(2)_{-1/2}$ WZW is well-defined, one conventionally requires the construction of a modular invariant. This issue, however, is plagued with various difficulties, as we discuss.

1 Introduction

In a previous work [1], we studied the $\beta\gamma$ system and its relations with the $\widehat{su}(2)_{-1/2}$ WZW model.

The $\beta\gamma$ system is a simple and crucial ingredient in the description of phase transitions in disordered electronic metals, 2D gravity, and string theory. While a 'free' theory, a closer look at its physical properties reveals subtleties. In particular, we showed in [1] that in the $\widehat{su}(2)_{-1/2}$ incarnation, the spectrum of conformal weights is not bounded from below, and operators with increasingly negative dimensions appear. This complexity is, however, smoothed out by the fact that all the operators can be organized in terms of four families, labeled by the four admissible representations of the $\widehat{su}(2)_{-1/2}$ algebra [2].

The infinite number of representations within a family are related to each other through the spectral-flow symmetry. Viewed from the perspective of the $\beta\gamma$ system, these flowed representations are associated to what we called deeper twists [1].

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A substantial portion of our previous work was devoted to putting the equivalence between the $\beta\gamma$ system and the $\widehat{su}(2)_{-1/2}$ WZW model on a firm basis. In particular, we compared the correlators of the ghost theory with the solutions to the Knizhnik-Zamolodchikov equation. The faithful free-field representation of the $\beta\gamma$ system, in terms of the $\eta\xi$ ghosts (with $c=-2$) and a Lorentzian boson, played a crucial role in our analysis of the correlators.

The free-field representation also allowed us to make definite statements concerning the fusion rules of the model. In that regard, comparing our conclusions with those of another recent work devoted to the analysis of the $\widehat{su}(2)_{-4/3}$ WZW model [3], the results may appear puzzling. It was found there that continuous and indecomposable (or logarithmic) representations necessarily appear in fusion rules. In contrast, we found that the $k=-1/2$ fusion rules close within the set of usual highest-weight representations and their flowed versions (which include the corresponding lowest-weight representations). In other words, neither continuous nor indecomposable representations are enforced by closure of the $\widehat{su}(2)_{-1/2}$ fusion algebra.

The bottom line of this discrepancy is that the $\widehat{su}(2)_{-4/3}$ WZW model is believed to be a logarithmic conformal field theory [3], while the $\widehat{su}(2)_{-1/2}$ WZW model in its $\beta\gamma$ formulation appears to be a quasi-rational conformal field theory [1]. Since the level is admissible in both cases and since admissible $\widehat{su}(2)$ models can be expected to behave similarly, such a different behavior is surprising.

Several resolutions seem to suggest themselves, though. First, an immediate and quite natural interpretation of the discrepancy is that the theory could have various formulations, that is, various *lifts*, and that we are actually not comparing the same version of the two models.

Let us clarify this idea of 'various formulations' within the precise context of the $k=-1/2$ model. The free-field representation used in [1] is (see also section 3 for a more detailed discussion of this representation):

$$\beta = e^{-i\phi}\eta, \qquad \gamma = e^{i\phi}\partial\xi\ , \tag{1}$$

where η and ξ are the $c=-2$ fermionic ghosts of weight $h=1$ and $h=0$, respectively. The point we want to stress is that the free-field representation does not take advantage of the full $\eta\xi$ algebra; it only uses the algebra spanned by η and $\partial\xi$, the so-called *small algebra* [4]. In other words, we have not used the zero mode of ξ. It is thus natural to investigate the effect, on the spectrum, of introducing this zero mode. Adding a zero mode to the theory is what we refer to loosely as a *lift*. Here it is a lift from the small to the *large algebra*. Although such a lift does not affect the central charge, it certainly generates additional states.

Actually, the extra zero mode is responsible for the occurrence of new *continuous* representations. An analogous example is the representation denoted E in [3]. Here we show that, in the more general case, they correspond collectively to the so-called *relaxed representations* of [5] (whose structure is briefly reviewed in the appendix).

But this lifting process may be pushed a step further. If we extend the dimension-one ghost η by a zero mode, more precisely the zero mode of $\partial^{-1}\eta$, one would then have a further lift. This corresponds to a representation in terms of the $c=-2$ symplectic fermion theory or *symplectic algebra*. At this stage, there are thus two zero modes and the corresponding $c=-2$ theory is known to be logarithmic. This characteristic will obviously be shared by the parent $c=-1$ model. The symplectic formulation thus leads to a candidate logarithmic version of the $\widehat{su}(2)_{-1/2}$ WZW model.

In fact, by extending the theory by two fermionic zero-modes, it is clear that every bosonic state can thereby be paired with a bosonic partner. The latter is obtained simply by acting upon the original state with the product of these two fermionic zero modes. This is a candidate for a two-dimensional Jordan cell – an indicator of a logarithmic theory. This contention is indeed confirmed and we find that the resulting theory is now very similar to that analyzed in [3]. In particular, we see the indecomposable representations emerging and the rather complicated pattern of their 'extremal diagram' is made very concrete by our free-field computations. In this context, we make the new observation that the indecomposable representations have two constituent relaxed modules.

A natural question at this stage is the following: are these various versions of the $\widehat{su}(2)_{-1/2}$ WZW model all well-defined theories or, say, is the logarithmic version the only viable one (as the analysis of [3] would suggest)? In all versions, we show that there is a closed fusion algebra. However, one may be

dealing merely with a closed *subalgebra* at each step of some 'larger' theory. For the Ising model, for instance, the fusion rules close within the Neveu-Schwarz sector. However, the physical model has to include the Ramond sector as well. In such a circumstance, the presence of extra fields becomes apparent at the level of the construction of the modular invariant. Therefore, these questions point toward the analysis of the modular invariant partition function. Is a modular invariant meaningful for any of these versions (i.e., can we construct three different modular invariants for the $\widehat{su}(2)_{-1/2}$ WZW model) or is there a single 'master' modular invariant associated to the logarithm version of the theory?

Somewhat surprisingly, the modular invariant issue is plagued with various difficulties. Although we do not propose definite answers, some of the subtleties related to this problem are pointed out in section 5.

The paper is organized as follows. First we discuss purely algebraic structures of the $\widehat{su}(2)_{-1/2}$ algebra without reference to the free fields. We discuss the spectral flow symmetry and the different standard modules. The extension to relaxed modules and some of their embedding patterns is presented in the appendix, which can be read after section 2. We then analyze in turn the extension of the theory from the small to the large algebra (section 3) and then its extension to the symplectic formulation (section 4). The problems linked to the interpretation of the modular invariant are addressed in section 5.

Notation:
For an easy reference, we summarize our notation for the various types of modules used here (which will be defined in due time):

M: highest-weight Verma module;
$M^\star$: irreducible highest-weight module;
R: relaxed module;
$R^\star$: irreducible relaxed module;
R^{a}: almost reduced relaxed module;
$\mathcal{I}$: indecomposable module.

2 Admissible modules, flows and fusion

In this section, we review results concerning the spectral flow and the related twisted modules or representations. Some of these results were already reported in [1] but here the emphasis is placed somewhat differently. Moreover, we will attempt to place our results within a more general algebraic framework discussed in particular in [5, 6]. For this, it will be convenient to slightly modify our previous notation and adjust it to the ones of these references. After this short review, we relate the spectral flow to a symmetry of the fusion rules. We then provide a simple characterization of those models for which all fusion rules can be obtained by flows of the fusion of level-zero finite-dimensional representations. This provides a simple distinction between the level $-1/2$ and the level $-4/3$ cases.

2.1 Spectral flow and twisted modules

The affine Lie algebra $\widehat{su}(2)_k$ with level k is defined by the commutator relations

$$\begin{aligned} \left[J_m^+, J_n^-\right] &= 2J^3_{m+n} + km\delta_{m+n,0} \ , \\ \left[J^3_m, J^\pm_n\right] &= \pm J^\pm_{m+n} \ , \\ \left[J^3_m, J^3_n\right] &= \frac{k}{2}m\delta_{m+n,0} \ . \end{aligned} \tag{2}$$

The spectral flow [7, 5, 6], with flow parameter[1] $\theta \in \mathbb{Z}$, is the algebra automorphism

$$\pi_\theta \ : \qquad J_n^\pm \ \mapsto \ J^\pm_{n\pm\theta}, \qquad J^3_n \ \mapsto \ J^3_n + \frac{k}{2}\theta\delta_{n,0} \ . \tag{3}$$

[1] In our previous work [1] we used the inverse flow, acting as $\pi_w : \ J_n^+ \mapsto J^+_{n-w}$, for example.

A twisted Verma module, $M_{j,t;\theta}$, is freely generated by $J^+_{n\leq\theta-1}$, $J^3_{n\leq-1}$, and $J^-_{n\leq-\theta}$ from the twisted highest-weight vector $|j,t;\theta\rangle$ defined by the conditions

$$
\begin{aligned}
J^+_{n\geq\theta}|j,t;\theta\rangle \;=\; J^3_{n\geq 1}|j,t;\theta\rangle \;=\; J^-_{n\geq 1-\theta}|j,t;\theta\rangle \;&=\; 0 \;, \\
\left(J^3_0+\frac{k}{2}\theta\right)|j,t;\theta\rangle \;&=\; j|j,t;\theta\rangle \;.
\end{aligned}
\tag{4}
$$

j is referred to as the spin of the vector $|j,t;\theta\rangle$. In order to comply with the notation of [6] we use $t\equiv k+2$ in the characterization of the module and its vectors. We assume $t\neq 0$. When the twist parameter $\theta\in\mathbb{Z}$ is omitted, it is understood to be zero and the module or vector is not twisted.

A vector in the twisted module $M_{j,t;\theta}$ can be assigned the level N if it can be written as a linear combination of monomials

$$
J^{a_1}_{-n_1}\dots J^{a_i}_{-n_i}|j,t;\theta\rangle \;, \tag{5}
$$

having $\sum_{j=1}^{i} n_j=N$. Note that some n_j may be negative. Such a vector has J^3_0 eigenvalue of the form

$$
m\in\left(j-\frac{k}{2}\theta+\mathbb{Z}\right) \;. \tag{6}
$$

This eigenvalue is called the charge of the vector. Within a module, we choose to talk about the level of a vector instead of its conformal weight, i.e., its L_0 eigenvalue where L_0 is the zero mode of the Sugawara energy-momentum tensor T. The rationale for this is that the notion of level is still well-defined for indecomposable representations, to be discussed below, while the action of L_0 generally is non-diagonal.

The vectors in $M_{j,t;\theta}$ (as well as in the more complicated indecomposable modules) can always be organized in linear combinations having definite charge and level. The representation of $M_{j,t;\theta}$ by a semi-infinite square lattice in the $(charge,-level)$ plane is called a diagram (we used similar diagrams, albeit with different conventions, in Fig. 1 of our previous paper [1]). As indicated, the second axis is commonly inverted. A vertex represents the finitely many vectors with given $(m,-N)$ values. From (6) and the definition of level, it follows that the lattice is integer spaced, and shifted from the integer points by the vector $(j-\frac{k}{2}\theta,0)$. It is natural to make an additional shift by $(0,h_{j,\theta})$, where $h_{j,\theta}$ is the conformal dimension of the twisted highest-weight state $|j,t;\theta\rangle$:

$$
h_{j,\theta}=\frac{j(j+1)}{k+2}-j\theta+\frac{k}{4}\theta^2 \;. \tag{7}
$$

Vertices are connected by arrows if the corresponding vectors can be reached by the action of the $\widehat{su}(2)_k$ generators. The extremal diagram is the boundary of the lattice and is often the only part illustrated.

With its action extended to Verma modules, π induces the following translation

$$
\pi_\theta(M_{j,t;\theta'})=M_{j,t;\theta+\theta'} \;. \tag{8}
$$

The flow and twist parameters are then identified. It follows from (8) that (twisted) Verma modules may be organized in orbits under the spectral flow. Under the action of π_1, a highest-weight module $M_{j,t}$ is sent to a lowest-weight (that is, lowest weight with respect to the finite $su(2)$ algebra acting at level zero) module $M_{j,t;1}$, see Fig. 1. Thus, in every orbit there are at least one highest-weight and one lowest-weight module. As we will discuss shortly, some modules are both highest-weight and lowest-weight modules.

2.2 Reducible modules and admissible representations

A singular vector exists in $M_{j,t}$ if and only if the spin j takes one of the two forms

$$
\begin{aligned}
j=j^+(r,s,t)&=\frac{r-1}{2}-\frac{s-1}{2}t, \qquad r,s\in\mathbb{N}\;, \\
j=j^-(r,s,t)&=-\frac{r+1}{2}+\frac{s}{2}t, \qquad r,s\in\mathbb{N}\;.
\end{aligned}
\tag{9}
$$

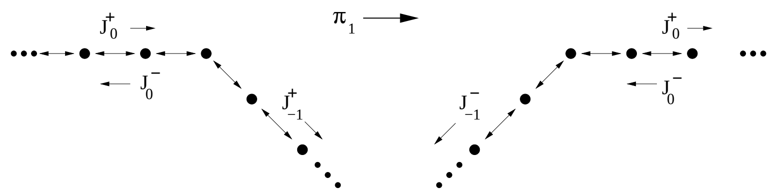

Figure 1: Diagram representations of $M_{j,t;\theta}$, where the horizontal axis is the charge, the vertical axis the level. The 'extremal diagram' on the left represents an untwisted highest-weight module. The corresponding (highest-weight) $su(2)$ representation at level zero may extend infinitely towards the left. π_1 maps this module onto a lowest-weight module, represented by the diagram on the right.

The singular vectors in $M_{j^{\pm}(r,s,t),t}$ have been worked out explicitly by Malikov, Feigin and Fuchs [8] and are referred to as MFF vectors. They appear at levels $r(s-1)$ and rs, respectively.

Due to the presence of singular vectors, some modules are reducible. Their irreducible counterparts are obtained by factoring out the submodules generated from the singular vectors. At most two of the MFF vectors are primitive, meaning that we need to factor out vectors generated from at most two singular vectors. The irreducible module associated to the Verma module $M_{j,t}$ will be denoted $M^{\star}_{j,t}$.

For fractional or integer $t = k+2 > 0$

$$t = p/p', \qquad (p,p') = 1 \ , \tag{10}$$

the admissible spins are of the form j^+ appearing in (9) and subject to the constraints

$$j(r,s) = \frac{r-1}{2} - \frac{s-1}{2}t, \qquad 1 \leq r \leq p-1, \quad 1 \leq s \leq p' \ . \tag{11}$$

For the case $t = 3/2$ ($k = -1/2$) of our previous paper [1], we have $p = 3$ and $p' = 2$. This corresponds to the $\beta\gamma$ system, and there are thus four admissible spins:

$$j(1,1) = 0, \qquad j(2,1) = 1/2, \qquad j(2,2) = -1/4, \qquad j(1,2) = -3/4 \ . \tag{12}$$

For generic t, the total number of admissible spins is $(p-1)p'$.

A representation is called admissible if it corresponds to a highest-weight module, $M_{j,t}$, with j an admissible spin. The horizontal extremal diagram, i.e., the $su(2)$ module generated by the action of the zero modes on $|j,t\rangle$, is finite-dimensional if and only if $s = 1$ (it is otherwise semi-infinite). An irreducible Verma module $M^{\star}_{j(r,1),t}$ may be regarded as a highest-weight and a lowest-weight module simultaneously. We sometimes loosely call the $p-1$ modules $M^{\star}_{j(r,1),t}$ finite-dimensional, even though this designation applies to their horizontal extremal diagrams only. We call an orbit containing an admissible highest-weight module an admissible orbit.

As already mentioned, $\pi_1(M^{\star}_{j(r,s),t})$ is a lowest-weight module. In order for it to be a highest-weight module as well, the vector $(J^+_{-1})^u|j(r,s),t\rangle$ in the extremal diagram of $M_{j(r,s),t}$ must be singular (see Fig. 1). This is seen to require

$$u = k - 2j + 1 = -r + st \ , \tag{13}$$

and since u is a positive integer, we find that $s = p'$. In terms of reduced modules, we conclude that the only orbits with more than one highest-weight module are the ones containing the finite-dimensional modules. For fractional level[2], $p' > 1$, these orbits contain exactly two highest-weight modules, and the latter are related as

$$\pi_{-1}(M^{\star}_{j(r,1),t}) = M^{\star}_{j(r,p'),t} \ . \tag{14}$$

In particular, two finite-dimensional modules cannot belong to the same orbit. A simple counting shows that there are $(p-1)(p'-1)$ admissible orbits (in the case with $p = 3$ and $p' = 2$ there are thus two

[2] We are not concerned with the simpler case of integer levels (where $p' = 1$).

admissible orbits). We note that the number of finite-dimensional modules is equal to the number of admissible orbits if and only if $p' = 2$. This is of importance for the closure of the fusion algebra to be discussed in the following subsection.

2.3 Closure of the fusion algebra

For simplicity, when discussing fusions we shall label the fused fields by their associated modules, e.g., we may consider the fusion $M^\star_{j,t;\theta} \times M^\star_{j',t}$. Following [3, 1], we assume that fusion products are invariant under the spectral flow:

$$\pi_\theta(M^\star_1) \times \pi_{\theta'}(M^\star_2) = \pi_{\theta+\theta'}(M^\star_1 \times M^\star_2) \tag{15}$$

for some irreducible modules $M^\star_i$. Later on we shall extend this assumption to include relaxed and indecomposable modules as well. It follows from (15) that the fusion product of two modules is determined by the fusion product of the highest-weight modules in their respective orbits.

Let us now consider the case where fusion closes on the set of finite-dimensional modules or representations, i.e., on the set of admissible irreducible modules $M^\star_{j(r,1),t}$. In the integer-level case, this is certainly always the case, while in the fractional-level case it remains an assumption in general. However, for $\widehat{su}(2)_{-1/2}$, this has been demonstrated explicitly using a free-field realization [1].

Now, whenever fusion closes on the set of finite-dimensional modules (and as we just said, this is so for $k = -1/2$) and for all cases where $p' = 2$ (which includes $k = -1/2$), it follows that the complete fusion algebra is fixed by the fusion algebra of the finite-dimensional representations. Closure of the fusion algebra is thereby ensured (as for $k = -1/2$, for example). On the other hand, we also see that the fusion algebra is not fully determined by the fusion algebra of the finite-dimensional representations when $p' > 2$. It may still close, i.e., close on the set of $M^\star_{j,t;\theta}$, but the analysis of the $k = -4/3$ example in [3] suggests the contrary. We find it thus natural to conjecture that the fusion algebra closes on the set of admissible representations and their flowed companions if and only if $p' = 2$.

For $p' = 2$, it turns out that the fusion algebra of twisted modules, $M^\star_{j(r,s),p/2;\theta}$, closes on the subset defined by $\theta \in \mathbb{Z}_\geq$ (or the subset defined by $\theta \in \mathbb{Z}_\leq$) as well. Of course, closure of the fusion algebra is only a necessary requirement for a well-defined theory; among other requirements, modular invariance of the partition function must be addressed. We get back to this point in section 5.

2.4 Characters of representations as series expansions of character functions

Let us now describe the irreducible characters of the admissible representations and the associated concept of character functions.

To each twisted admissible representation there is an associated character, $\chi_{j(r,s),t;\theta}$, defined by

$$\chi_{j(r,s),t;\theta}(z,q) = \mathrm{Tr}_{M^\star_{j(r,s),t;\theta}} q^{L_0 - c/24} z^{J^3_0}\ , \tag{16}$$

where z and q are formal parameters. In the following we will consider functions of z and q and assume that $|q| < 1$, allowing us to consider the constraint as a q-dependent region in the z plane. In order to comply with the notation of [9], we have changed our notation from y in [1] to $z = y^2$. We stress that the characters (16) are formal series in the parameters q and z. As discussed in [9], these series only converge in certain domains (annuli) of the complex plane, while they can be continued outside. The resulting meromorphic functions are called character functions, and must not be confused with the original characters. We will denote these character functions $F_{j(r,s),t;\theta}(z,q)$.[3]

We stress that $F_{j(r,s),t;\theta}(z,q)$ is holomorphic if and only if $s = p' = 1$, i.e., for integrable representations.

[3] To simplify the notation, we will often use the notation $F_{j(r,s)}(z,q) = F_{j(r,s),\frac{3}{2};0}(z,q)$ for the four distinct character functions in the case $k = -1/2$.

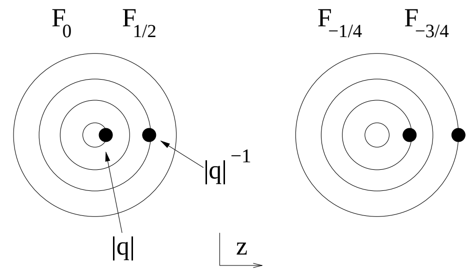

Figure 2: Schematic description of the distribution of poles.

The functions $F_{j(r,s),t;\theta}(z,q)$ may be organized in finitely many orbits. To see this, we trivially extend the definition of the spectral flow to the character functions as

$$\pi_{\theta'}(F_{j(r,s),t;\theta})(z,q) = F_{j(r,s),t;\theta+\theta'}(z,q) \ , \tag{17}$$

where

$$F_{j,t;\theta}(z,q) = q^{\frac{t-2}{4}\theta^2} z^{-\frac{t-2}{2}\theta} F_{j,t;0}(zq^{-\theta},q) \ . \tag{18}$$

Using the relations (cf. [9], Theorem 4.2)

$$\begin{aligned} \pi_{2p'}(F_{j(r,s),t;\theta})(z,q) &= F_{j(r,s),t;\theta}(z,q) \ , \\ \pi_{p'}(F_{j(r,s),t;\theta})(z,q) &= (-1)^{p'-1} F_{j(p-r,s),t;\theta}(z,q) \ , \end{aligned} \tag{19}$$

one observes that only a finite number of functions are generated under the spectral flow.[4] In the case with $k = -1/2$, where $t = p/p' = 3/2$, the relations (19) lead to a periodicity four in the set of character functions. This periodicity has also been noticed in section 6.3 of [1].

The character functions are meromorphic in z; their singularities are all simple poles. The distribution of these poles is as follows: for $j = j(r,s) = j^+(r,s,t)$, the function $F_j(z,q)$ has poles at the points (cf. [9], Lemma 4.4)

$$z = q^n, \qquad n \in \mathbb{Z} \setminus (p'\mathbb{Z} + s - 1) \ . \tag{20}$$

For $p' = 2$, this leads to a pole on every second circle defined by $|z| = |q^n|$. The pole structure of the four character functions appearing for $k = -1/2$ is illustrated in Fig. 2. Let us describe it in more details. For $s = 1$ we have a pole located at every odd power of q. Consider for instance the character of the untwisted ($\theta = 0$) identity representation with $j(1,1) = 0$. The series expansion was originally defined in the annulus $1 < |z| < |q|^{-1}$. However, since there is no pole on the unit circle, this expansion is extended to an open 'double annulus' $|q| < |z| < |q|^{-1}$.[5] The character of the untwisted identity representation thus reads

$$\chi_{j(1,1),\frac{3}{2};0}(z,q) = F_0(z,q) \qquad \text{iff} \quad |q| < |z| < |q|^{-1} \ , \tag{21}$$

with

$$F_0 = q^{1/24} \frac{\sum_{m\in Z} q^{6m^2-2m} z^{-3m} - z^{-1} \sum_{m\in Z} q^{6m^2+2m} z^{-3m}}{\prod_{m\geq 0}(1 - z^{-1}q^m) \prod_{m\geq 1}(1 - zq^m)(1 - q^m)} \ . \tag{22}$$

A similar result holds for the spin-1/2 representation with

$$F_{1/2} = q^{1/24} q^{1/2} z^{1/2} \frac{\sum_{m\in Z} q^{6m^2-4m} z^{-3m} - z^{-2} \sum_{m\in Z} q^{6m^2+4m} z^{-3m}}{\prod_{m\geq 0}(1 - z^{-1}q^m) \prod_{m\geq 1}(1 - zq^m)(1 - q^m)} \ . \tag{23}$$

[4] There are additional relations when $p = 2r$ (see [9] for details).

[5] We stress that the extension of the convergence domain to two basic annuli is a special feature of the $p' = 2$ models. The situation for all fractional levels with $p' = 2$ is easily described: poles appear exactly after every second annulus. The generalization of Fig. 2 is straightforward: the $p - 1$ character functions associated to finite-dimensional representations, $j(r,1)$, have identical distributions (extending the left part of the figure), while the $p-1$ infinite-dimensional representations, $j(r,p')$, give rise to the functions with distribution as to the right in Fig. 2.

In the same way, the character of the $j(2,2) = -1/4$ representation is found to be

$$\chi_{j(2,2),\frac{3}{2};0}(z,q) = F_{-1/4}(z,q) \qquad \text{iff} \;\; 1 < |z| < |q|^{-2} \;, \tag{24}$$

with

$$F_{-1/4} = q^{1/24} q^{-1/8} z^{-1/4} \frac{\sum_{m\in Z} q^{6m^2-m} z^{-3m} - z^{-2}q^2 \sum_{m\in Z} q^{6m^2+7m} z^{-3m}}{\prod_{m\geq 0}(1-z^{-1}q^m)\prod_{m\geq 1}(1-zq^m)(1-q^m)} \;, \tag{25}$$

and similarly for the spin $j = -3/4$ with

$$F_{-3/4} = q^{1/24} q^{-1/8} z^{-3/4} \frac{\sum_{m\in Z} q^{6m^2+m} z^{-3m} - z^{-1}q \sum_{m\in Z} q^{6m^2+5m} z^{-3m}}{\prod_{m\geq 0}(1-z^{-1}q^m)\prod_{m\geq 1}(1-zq^m)(1-q^m)} \;. \tag{26}$$

Consistency with the spectral flow follows from simple identities between the character functions. One such identity is

$$\pi_{-1}(F_{j(1,1),\frac{3}{2};0})(z,q) = F_{j(1,1),\frac{3}{2};-1}(z,q) = q^{-\frac{1}{8}} z^{-\frac{1}{4}} F_0(zq,q) = F_{-1/4}(z,q) \;. \tag{27}$$

Note that both functions can be expanded to the desired characters only in the annulus $1 < |z| < |q|^{-1}$.

The presence of poles in highest-weight representations with $s \neq 1$ is rooted in that at level zero there is an infinite number of states simply because the representation is not also lowest weight. That explains, for instance, the fact that $F_{-1/4}$ has a pole at $z = 1$. But how can we see the poles in F_0, say? For the vacuum representation there is no pole at $z = 1$ because the number of states at each level is finite, as for an integrable representation. However, it differs from an integrable representation in being 'wider' at each level. In particular, we have $(J^-_{-1})^n I \neq 0$ for any n (i.e., on the 45^o N-W diagonal there are states at every point – in sharp contrast to an integrable representation). This implies that if we fix $z = q$, whose effect is that all states on this diagonal are sent to level zero, the character becomes infinite.

The divergence of the characters corresponds to a divergence of the functional integral defining the $\beta\gamma$ system (see our first paper [1]) when a field is coupled to the $U(1)$ charge. Analytic continuation beyond the circles of convergence can be seen as a regularization of this integral. This will be discussed more in section 5.

3 The $\widehat{su}(2)_{-1/2}$ model revisited

In the following sections we concentrate on the $\widehat{su}(2)_{-1/2}$ model. We discuss free-field realizations of the small and large $c = -2$ algebras. To keep the notation simple, we leave out the dependency on $t = k+2 = 3/2$, as in $M_{j;\theta} = M_{j,3/2;\theta}$, for example.

3.1 Free-field realization of the small algebra

In our previous paper [1] we discussed the $\beta\gamma$ system, described by the energy-momentum tensor[6]

$$T = \frac{1}{2}(\beta\partial\gamma - \partial\beta\gamma) \tag{28}$$

in the left-moving sector. In this formulation, β and γ have weight $h = 1/2$ and charge $1/2$ and $-1/2$, respectively, with respect to the current

$$J^3 = -\frac{1}{2}\gamma\beta \;. \tag{29}$$

[6]Here and in the following we omit normal-ordering signs whenever confusion is unlikely to occur.

The full $\widehat{su}(2)$ symmetry emerges through the construction of the currents as

$$\begin{aligned} J^+ &= \frac{1}{2}\beta^2 \ , \\ J^3 &= -\frac{1}{2}\gamma\beta \ , \\ J^- &= -\frac{1}{2}\gamma^2 \ . \end{aligned} \tag{30}$$

A more convenient representation of these currents is obtained by introducing

$$\beta = e^{-i\phi}\eta, \quad \gamma = e^{i\phi}\partial\xi \ , \tag{31}$$

where ϕ is a free boson with negative metric

$$\langle\phi(z)\phi(w)\rangle = \ln(z-w) \ , \tag{32}$$

while η and ξ are fermions of weight $h = 1$ and $h = 0$, respectively, satisfying

$$\langle\eta(z)\xi(w)\rangle = \frac{1}{z-w} \ . \tag{33}$$

In this representation, we have[7]

$$\begin{aligned} J^+ &= \frac{1}{2}e^{-2i\phi}\partial\eta\eta \ , \\ J^3 &= \frac{i}{2}\partial\phi \ , \\ J^- &= -\frac{1}{2}e^{2i\phi}\partial^2\xi\partial\xi \ . \end{aligned} \tag{34}$$

This free-field representation provides a faithful description of the relevant modules in the theory. For example, the irreducible modules $M^\star_{0;\theta}$ and $M^\star_{1/2;\theta}$ for small values of $|\theta|$ are represented by the components [1]:

$$\begin{aligned} M^\star_{0;0} &: \quad \{1\} \ , \\ M^\star_{0;1} &: \quad \{e^{-i\phi/2}\} \ , \\ M^\star_{0;-1} = M^\star_{-1/4;0} &: \quad \{e^{i\phi/2}\} \ , \\ M^\star_{1/2;0} &: \quad \{e^{-i\phi}\eta\} \ , \\ M^\star_{1/2;1} &: \quad \{e^{-3i\phi/2}\eta\} \ , \\ M^\star_{1/2;-1} = M^\star_{-3/4;0} &: \quad \{e^{3i\phi/2}\partial\xi\} \ . \end{aligned} \tag{35}$$

Any vector in the irreducible module is generated from the above representative by the action of the J^a_n modes as given by their free-field expressions. The resulting modules may be identified with well-known representations. In the same order as listed above, and in the notation of [1], we have D_0, $D^-_{-1/4}$, $D^+_{-1/4}$, $D_{1/2}$, $D^-_{-3/4}$, and $D^+_{-3/4}$.

In this free-field picture, the action of the spectral flow amounts to a multiplication by a vertex operator:

$$\pi_\theta(\Psi) \ \rightarrow \ :\Psi e^{-i\theta\phi/2}: \ . \tag{36}$$

It is simple to verify that the fusion algebra of all these fields closes.

The currents J^a involve the field ξ only through its derivative, and this is also true for the representations described above, (35): in the terminology of [4, 10], we are working with the small algebra.

[7] The minus sign in J^- corrects a misprint in [1].

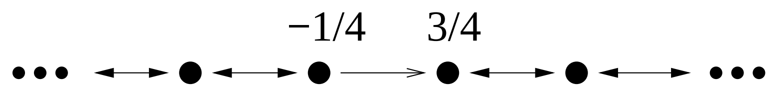

Figure 3: Extremal diagram for the relaxed module generated from $\xi e^{i\phi/2}$.

3.2 Extension to the large algebra

Extending the theory by including ξ amounts to studying the large algebra instead of the small algebra discussed above. By this extension, the possibility of new and possibly reducible modules emerges. In order to see that such modules actually appear, let us study the representative

$$\xi e^{i\phi/2} \ . \tag{37}$$

By acting with J^+ we find

$$\begin{aligned} J^+(z)\ (e^{i\phi/2}\xi)(w) &= \frac{1}{2}e^{-2i\phi}\partial\eta\eta(z)\ e^{i\phi/2}\xi(w) \\ &= \frac{1}{(z-w)}\left(\frac{1}{2}e^{-3i\phi/2}\eta\right)(w) + \cdots \ , \end{aligned} \tag{38}$$

and similarly

$$\begin{aligned} J^-(z)\ (e^{i\phi/2}\xi)(w) &= -\frac{1}{2}e^{2i\phi}\partial^2\xi\partial\xi(z)\ e^{i\phi/2}\xi(w) \\ &= \frac{1}{(z-w)}\left(-\frac{1}{2}e^{5i\phi/2}\partial^2\xi\partial\xi\xi\right)(w) + \cdots \ . \end{aligned} \tag{39}$$

Here and in the following we shall make frequent use of the expansions

$$\begin{aligned} J^a(z) &= \sum_{n\in\mathbb{Z}} \frac{J^a_n(w)}{(z-w)^{n+1}} \ , \\ T(z) &= \sum_{n\in\mathbb{Z}} \frac{L_n(w)}{(z-w)^{n+2}} \ , \end{aligned} \tag{40}$$

around the point w of the affine currents and the energy-momentum tensor. In particular, we see from (38) that

$$J^+_0(e^{i\phi/2}\xi) = \frac{1}{2}e^{-3i\phi/2}\eta \ . \tag{41}$$

On the other hand, this last expression corresponds to the lowest-weight state with J^3_0 eigenvalue 3/4 of the lowest-weight representation $M^*_{1/2;1}$. The extremal diagram of the full representation, i.e., the one obtained by acting with the current generators on $\xi e^{i\phi/2}$, is depicted in Fig. 3. (Note that it differs from Fig. 7 in the appendix.)

As shown in Fig. 3, the representation is reducible. The Verma module associated with this representation is that of a relaxed module containing a charged singular vector in the terminology of [5, 6].[8]

Let us pause to explain in simple terms the essence of relaxed and almost reduced modules (formal definitions can be found in the appendix). An untwisted relaxed module, denoted R, is an affine Lie algebra highest-weight module except for the fact that the J^+_0 highest-weight condition is no longer imposed. As a result, if j is not half-integer (meaning that the level-zero $su(2)$ representation is not

[8] For example, using the notation introduced in the appendix, we can label the vector with eigenvalue $J^3_0 = -1/4$ by $|\mu_1, \mu_2; \theta\rangle = |0, -1/2; 0\rangle$.

lowest-weight, in particular), the representation at level zero is infinite in both directions. A relaxed module may contain singular vectors in which case the corresponding irreducible module (written $R^\star$) is obtained by factoring out these singular vectors. In a relaxed module, there may be a (special) singular vector at level zero. It is called a charged singular vector. If all but this special singular vector are factored out, the module is said to be almost reduced and is denoted R^{a}. Relaxed modules without charged singular vectors are discussed in [3, 1] and are there denoted E and are referred to as continuous representations (see also the appendix). As for ordinary modules, relaxed modules may be twisted.

We now compare the large-algebra module just obtained (and generated from (37)) with those generated by the small algebra. The main difference is that this new module, described by the free fields in the large algebra, is *not* the irreducible module obtained by removing all singular vectors. Rather, it is a reducible module obtained by removing all singular vectors but the charged singular one in the extremal diagram. The presence of this singular vector is signaled in Fig. 3 by the absence of an arrow going from $m = 3/4$ to $m = -1/4$. It is thus an example of an almost reduced module defined above.

We have discussed explicitly the structure of the module at the highest (extremal) level. Let us show that it has a similar pattern at lower levels as well. We start from the field $\xi e^{i\phi/2}$. By observing that the operator $e^{i\phi/2}$ is generating an irreducible $j = -1/4$ highest-weight representation, we know that the extremal diagram is such that $J^+_{-1} e^{i\phi/2} = 0$, for example. Similar conditions appear at lower levels. On the other hand, acting separately on ξ and $e^{i\phi/2}$, we observe that for every point where an annihilation condition is met for $e^{i\phi/2}$, one can contract one of the η's in J^+ with ξ to create a pole (or double pole) nullifying the extremal condition. Every time this is done, the zero mode of ξ is lost, and a similar arrow, as that at the top level, is created – we are now in a state that is part of the lowest-weight module with J^3_0 eigenvalues in $\mathbb{Z} + 3/4$.

We note that this extension to an almost reduced module is consistent with the 'equations of motion' since $C = -3/16$ (following from $|\mu_1 - \mu_2| = 1/2$, see (120)). From (36) it follows straightforwardly that ξ and $\xi e^{-i\phi/2}$, for example, correspond to states in modules[9], $R^{\mathrm{a}}_{0,-1/2;\theta}$, obtained under the spectral flow of $R^{\mathrm{a}}_{0,-1/2;0}$. Their characters are given by

$$\chi^{R^{\mathrm{a}}}_{0,-1/2;\theta}(z,q) = \delta(zq^{-\theta},1) Res_{zq^{-\theta}=1} \left[q^{-\theta^2/8} z^{\theta/4} F_{-1/4}(zq^{-\theta},q) \right] , \tag{42}$$

where the delta function is defined as follows

$$\delta(u,1) = \sum_{n\in\mathbb{Z}} u^n . \tag{43}$$

In the untwisted case, it reflects the fact that the spectrum of states at level zero is infinite in both directions. One finds

$$\chi^{R^{\mathrm{a}}}_{0,-1/2;\theta}(z,q) = \delta(zq^{-\theta},1) q^{-1/12} q^{-\theta^2/8} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} , \tag{44}$$

where we have used the identity[10]

$$\sum_{m\in Z} q^{6m^2-2m} q^{3m(2p+1)} - q^{2p+1} \sum_{m\in Z} q^{6m^2+2m} q^{3m(2p+1)} = q^{-p(3p+1)/2} \prod_{n=1}^{\infty} (1-q^n), \qquad (p\in\mathbb{Z}) . \tag{45}$$

There is, however, another module which is not in the orbit of $R^{\mathrm{a}}_{0,-1/2;0}$. It is generated from the representative

$$e^{3i\phi/2} \partial\xi\xi . \tag{46}$$

With an analysis similar to the previous one, we obtain the extremal diagram described in Fig. 4. The

[9] The notation is $R^{\mathrm{a}}_{\mu_1,\mu_2;\theta}$ where μ_1 and μ_2 are described in the appendix. The module $R^{\mathrm{a}}_{0,-1/2;0}$ is the one in Fig. 3.

[10] This identity can be proven, for instance, by some simple manipulations of the Jacobi θ_4 function.

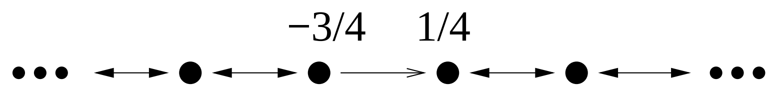

Figure 4: Extremal diagram for the relaxed module generated from $e^{3i\phi/2}\xi\partial\xi$.

characters of the almost reduced modules in the orbit of this one read

$$\chi_{0,1/2;\theta}^{R^{\mathrm{a}}}(z,q) = \delta(zq^{-\theta},1) Res_{zq^{-\theta}=1}\left[q^{-\theta^2/8}z^{\theta/4}F_{-3/4}(zq^{-\theta},q)\right] , \tag{47}$$

and one finds again

$$\chi_{0,1/2;\theta}^{R^{\mathrm{a}}}(z,q) = \delta(zq^{-\theta},1)q^{-1/12}q^{-\theta^2/8}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^2} . \tag{48}$$

In both examples, (42) and (47), we have chosen to characterize the relaxed module by the labels, μ_1 and μ_2, defining the neighbour vector to the left of the charged singular vector (see Fig. 3 and Fig. 4). This characterization is of course not unique, as every vector to the left of the charged singular one could play the role as the relaxed highest-weight vector. It is only in the fully reduced modules that these two choices are singled out.

The extension from the small to the large algebra is asymmetric since we chose to include the zero mode of ξ. The asymmetry shows up in the fact that only relaxed extensions of highest-weight representations are present. We could as well have introduced the zero mode of '$\partial^{-1}\eta$'. In the following, we shall discuss what happens when both are introduced.

Note that the theory we have obtained so far has a closed operator algebra, and does not involve indecomposable modules. We should therefore not expect to encounter logarithms in computations of correlators.

4 Logarithmic lift of the $\widehat{su}(2)_{-1/2}$ model

4.1 Symplectic fermions

Heuristically, the extension from the small to the large algebra above is done by keeping the states which were removed by the cohomology of ξ_0. As already mentioned, we could have studied the situation when keeping states removed by the cohomology of its companion instead. Here we want to consider the situation where no states are removed by the cohomology of a BRST operator. This leads to the introduction of 'symplectic fermions', ψ_1 and ψ_2, satisfying [10]

$$\psi_1(z)\psi_2(w) = -\ln(z-w) + :\psi_1(z)\psi_2(w): . \tag{49}$$

The extension is first done by integrating

$$\partial\psi_2 = \eta , \tag{50}$$

resulting in the mode expansion

$$\psi_2(z) = \alpha_2 + \eta_0 \log z - \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\eta_n z^{-n} , \tag{51}$$

thereby introducing an extra zero mode α_2. This leads to the conjugate field ψ_1 being a deformation of ξ

$$\psi_1 = \xi + \alpha_1 \log z , \tag{52}$$

with $\{\alpha_2,\alpha_1\} = 1$ and

$$\psi_1(z) = \xi_0 + \alpha_1 \log z + \sum_{n\in\mathbb{Z}\setminus\{0\}}\xi_n z^{-n} . \tag{53}$$

One observes that when restricting to the small algebra, generated here by $\partial\psi_1$ and $\partial\psi_2$, one has the same algebra and irreducible modules as we found in the first part of this paper.

The energy-momentum tensor is constructed by using the deformation performed in [11][11], or directly using the symplectic fermion construction:

$$\tilde{T}(z) = \partial\xi\eta(z) + \frac{\alpha_1\eta(z)}{z} = \partial\psi_1\partial\psi_2(z) \ . \tag{54}$$

The full energy-momentum tensor then reads

$$T = \frac{1}{2}\partial\phi\partial\phi + \partial\psi_1\partial\psi_2 \ . \tag{55}$$

Already at this level, we observe the vacuum Jordan cell ($\eta_0|\Omega\rangle = \alpha_1|\Omega\rangle = 0$),

$$\tilde{L}_0\alpha_2\xi_0|\Omega\rangle = |\Omega\rangle, \qquad \tilde{L}_0|\Omega\rangle = 0 \ . \tag{56}$$

Similarly, the generators of the $\widehat{su}(2)$ symmetry become to

$$\begin{aligned} J^+ &= \frac{1}{2}e^{-2i\phi}\partial^2\psi_2\partial\psi_2 \ , \\ J^3 &= \frac{i}{2}\partial\phi \ , \\ J^- &= -\frac{1}{2}e^{2i\phi}\partial^2\psi_1\partial\psi_1 \ . \end{aligned} \tag{57}$$

They still satisfy the same algebra, albeit the space on which they act is augmented by zero modes. In terms of the old variables, the above representation leads to the deformation of J^- only, and we have

$$J^- = \frac{1}{2}e^{2i\phi}\left(\partial^2\xi\partial\xi - \frac{\alpha_1\partial\xi}{z^2} - \frac{\alpha_1\partial^2\xi}{z}\right) \ . \tag{58}$$

Here we can describe the variable α_2 as the one enabling the creation of the new indecomposable modules. Essentially, the introduction of the zero mode creates an auxiliary space in which the larger affine representations are embedded. The zero mode α_1, on the other hand, acts trivially on the states not containing α_2 (i.e., $\alpha_1|\Omega\rangle = 0$ for example). On the states containing α_2 it allows the transition between the states in the auxiliary space.

It should be noted that the deformation presented above is not different from the one found in $c = -2$ theories, and therefore the results found there apply here as well. More interesting results are found when one studies the $su(2)$ indecomposable representations. This is what is done in the next sections.

The presence of indecomposable representations of $\widehat{su}(2)$ results in a logarithmic conformal field theory.

4.2 Indecomposable $j = 0$ module

Let us analyze the module generated from $|\omega\rangle$, the Jordan-cell partner to the vacuum state $|\Omega\rangle$. The corresponding fields are

$$|\omega\rangle \ \leftrightarrow \ \omega = \psi_2\psi_1, \qquad |\Omega\rangle \ \leftrightarrow \ \Omega = 1 \ . \tag{59}$$

From the OPE

$$T(z)\omega(w) = \frac{1}{(z-w)^2} + \frac{\partial\omega(w)}{(z-w)} + \mathcal{O}((z-w)^0) \ , \tag{60}$$

we immediately recover the Jordan-cell structure $L_0|\omega\rangle = |\Omega\rangle$. Due to the non-diagonal action of L_0, the module generated from $|\omega\rangle$ is indecomposable. We shall denote it $\mathcal{I}_0$ (where the $\mathcal{I}$ refers to it being indecomposable).

[11] A general study of how certain indecomposable modules may be obtained by considering short exact sequences involving extensions of ordinary modules, may be found in [11]. Their construction is not directly related to our results below, though.

To unravel the detailed structure of the module $\mathcal{I}_0$, we will evaluate various products of the currents with the field ω. Consider first

$$
\begin{aligned}
J^+(z)\omega(w) &= \frac{1}{2}\left(e^{-2i\phi}\partial^2\psi_2\partial\psi_2\right)(z)\,(\psi_2\psi_1)(w) \\
&= \frac{1}{(z-w)^2}\left(-\frac{1}{2}e^{-2i\phi}\partial\psi_2\psi_2\right) \\
&+ \frac{1}{z-w}\left(-e^{-2i\phi}\partial^2\psi_2\psi_2 + i\partial\phi e^{-2i\phi}\partial\psi_2\psi_2\right) \\
&+ (z-w)^0\left(\frac{1}{2}e^{-2i\phi}\partial^2\psi_2\partial\psi_2\psi_2\psi_1 - \frac{3}{4}e^{-2i\phi}\partial^3\psi_2\psi_2\right. \\
&\quad \left. + 2i\partial\phi e^{-2i\phi}\partial^2\psi_2\psi_2 + \frac{i}{2}\partial^2\phi e^{-2i\phi}\partial\psi_2\psi_2 + (\partial\phi)^2 e^{-2i\phi}\partial\psi_2\psi_2\right) \\
&+ \mathcal{O}(z-w)\ .
\end{aligned}
\tag{61}
$$

In the final expression, all arguments are evaluated at w. From this expansion, we read off the action of the various modes $J^+_{n\geq -1}$ on ω as

$$
\begin{aligned}
J^+_{n\geq 2}\omega &= 0\ , \\
J^+_1\omega &= -\frac{1}{2}e^{-2i\phi}\partial\psi_2\psi_2\ , \\
J^+_0\omega &= -e^{-2i\phi}\partial^2\psi_2\psi_2 + i\partial\phi e^{-2i\phi}\partial\psi_2\psi_2\ , \\
J^+_{-1}\omega &= \frac{1}{2}e^{-2i\phi}\partial^2\psi_2\partial\psi_2\psi_2\psi_1 - \frac{3}{4}e^{-2i\phi}\partial^3\psi_2\psi_2 + 2i\partial\phi e^{-2i\phi}\partial^2\psi_2\psi_2 \\
&\quad + \frac{i}{2}\partial^2\phi e^{-2i\phi}\partial\psi_2\psi_2 + (\partial\phi)^2 e^{-2i\phi}\partial\psi_2\psi_2\ .
\end{aligned}
\tag{62}
$$

The first observation to be made from these computations is that $|\omega\rangle$ is not an affine highest-weight state since $J^+_1\omega \neq 0$ (even though it is a Virasoro highest-weight state). Moreover, the action of the various modes J^+_n on ω do not produce fields that belong to the identity module: all the fields generated so far contain a ψ_2 without derivatives.

The extremal diagram of the module $\mathcal{I}_0$ is displayed in Fig. 5. The central black dot is associated to the ω field. So far, we have described the three arrows directed toward the right that leave the point ω, and these all connect other black dots. A black dot in Fig. 5 is always associated to a field containing either ψ_1 or ψ_2 (or both) without derivatives. All other fields in the module are represented by open dots.

Consider now the field $J^+_1\omega$. Acting on it with $J^+(z)$ gives

$$
J^+(z)(J^+_1\omega)(w) = \frac{1}{(z-w)^2}\left(-\frac{1}{8}e^{-4i\phi}\partial^3\psi_2\partial^2\psi_2\partial\psi_2\psi_2\right)(w) + \mathcal{O}((z-w)^{-1})\ , \tag{63}
$$

which shows that $(J^+_1)^2\omega \neq 0$. More generally, we find that

$$
(J^+_1)^{n+1}\omega = \left(\frac{-1}{2^{n+1}\prod_{m=1}^{2n} m!}\right) e^{-2(n+1)i\phi}\partial^{2n+1}\psi_2\ldots\partial^2\psi_2\partial\psi_2\psi_2\ . \tag{64}
$$

That $(J^+_1)^2\omega$ is non-vanishing accounts for the two N-E arrows in Fig. 5. Consider now the action of $J^-(z)$ on the field $(J^+_1)^2\omega$:

$$
J^-(z)(J^+_1)^2\omega(w) = \left(3e^{-2i\phi}\partial\psi_2\psi_2\right)(w) + \mathcal{O}(z-w)\ . \tag{65}
$$

The absence of a single pole implies that $J^-_0(J^+_1)^2\omega = 0$. From the first regular term, we infer that

$$
J^-_{-1}(J^+_1)^2\omega \ \propto\ J^+_1\omega\ . \tag{66}
$$

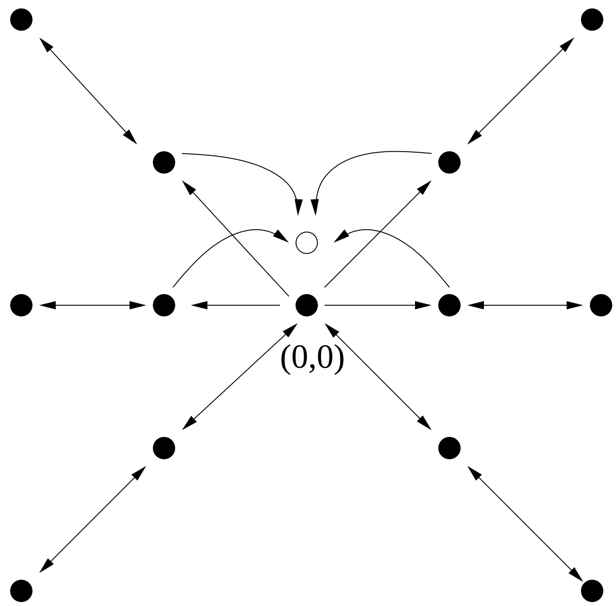

Figure 5: The indecomposable $j = 0$ module $\mathcal{I}_0$.

We have thus a double-sided arrow linking $J_1^+\omega$ and $(J_1^+)^2\omega$ in Fig. 5.

In the same way, let us consider the action of $J^-(z)$ on $(J_1^+\omega)(w)$:

$$J^-(z)(J_1^+\omega)(w) = (z-w)^0(1) + \mathcal{O}(z-w) \ . \tag{67}$$

This readily implies that $J_0^- J_1^+\omega = 0$. But more importantly, this OPE shows that the action of J_{-1}^- on $J_1^+\omega$ is not proportional to ω. Instead, it is proportional to the identity field. This reflects the indecomposability of the module. The arrow linking ω to $J_1^+\omega$ is thus not double-sided. There is an S-W arrow leaving $J_1^+\omega$ but it links it to the open dot representing the identity field, associated to $|\Omega\rangle$. The vectors in the module generated from $|\Omega\rangle$ (the highest-weight vector in the irreducible module $M_0^\star$) may be thought of as occupying integer (charge, level) lattice points ((m,h) with $h \geq |m|$) in a separate but equivalent layer underneath the layer limited by black dots in the downward part of the extremal diagram.

The other arrows in Fig. 5 are obtained analogously. We note the similarity of this module with the extended $j = 0$ module in the $\widehat{su}(2)_{-4/3}$ WZW model discussed in [3] (there denoted $\mathcal{R}_0$).[12]

As indicated in Fig. 5, the vectors $J_1^+|\omega\rangle$ and $J_1^-|\omega\rangle$ each generate a submodule in $\mathcal{I}_0$. Focusing on the module generated from the vector $J_1^+|\omega\rangle$, we immediately see that it corresponds to a π_1 twist of a module similar to the relaxed module with a charged singular vector in Fig. 4. Similarly, the module generated from the vector $J_1^-|\omega\rangle$ corresponds to a π_{-1} twist of a module similar to the relaxed module with a charged singular vector in Fig. 3. Note that both modules have the irreducible module $M_0^\star$, generated from $|\Omega\rangle$, as a submodule. This is simply due to the single-sided arrow linking $J_1^\pm\omega$ to $|\Omega\rangle$.

4.3 The $\mathcal{I}_0$ character

As discussed above, acting on the states $J_1^+\omega$ and $J_1^-\omega$ generates submodules that are twists of relaxed modules containing charged singular vectors, i.e., $\psi_2\partial\psi_2 e^{-2i\phi} = \pi_1(\psi_2\partial\psi_2 e^{-3i\phi/2})$ and $\psi_1\partial\psi_1 e^{2i\phi} = \pi_{-1}(\psi_1\partial\psi_1 e^{3i\phi/2})$. Since these states generate relaxed highest-weight modules in which the charged singular vector generates exactly the identity module, their irreducible module[13] $\chi^{R^\star}_{\mu_1,\mu_2;\theta}(z,q)$ will not

[12] There is a minor difference in the picture provided in [3], as the L_0 direction is inverted in that paper.

[13] Here we really consider the irreducible modules $R^\star$ from which the charged singular vector has been factored out.

count the identity module. In turn, the full indecomposable module will count each state in the identity module twice. It is not difficult to see that we have a character of the form

$$\begin{aligned} \chi^{\mathcal{I}}_{j=0}(z,q) &= 2\chi_{0;0}(z,q) + \chi_{-1,\frac{3}{2};1}(z,q) + \chi_{0,\frac{1}{2};-1}(z,q) \\ &= \chi^{R^a}_{-1,\frac{3}{2};1}(z,q) + \chi^{R^a}_{0,\frac{1}{2};-1}(z,q) \ , \end{aligned} \tag{68}$$

where each character is understood as a formal series. This appears to be a new observation.

4.4 Indecomposable $j = 1/2$ module

Here we demonstrate how the symplectic fermions admit an indecomposable extension, $\mathcal{I}_{1/2}$, of the irreducible $\widehat{su}(2)_{-1/2}$ spin-1/2 module $M^\star_{1/2}$. Proceeding in a way analogous to the construction of the $\mathcal{I}_0$ module above, we first need to identify the candidate Jordan-cell partners to the β and γ fields. We recall that the two ghost fields constitute an $su(2)$ spin-1/2 representation. It is natural to expect that each of these fields is to be combined with its ω-composite. We first focus on β, in which case we thus expect the pair β and

$$\beta\omega = e^{-i\phi}\partial\psi_2\psi_2\psi_1 \ , \tag{69}$$

to form a Jordan cell. Let us check this by computing the OPE of T with $\beta\omega$:

$$T(z)\beta\omega(w) = \frac{(-e^{-i\phi}\psi_2)(w)}{(z-w)^3} + \frac{(\beta + \frac{1}{2}\beta\omega)(w)}{(z-w)^2} + \frac{(\partial(\beta\omega))(w)}{(z-w)} + \mathcal{O}((z-w)^0) \ . \tag{70}$$

This shows that our candidate field $\beta\omega$ is indeed the Jordan-cell partner to β since

$$L_0|\beta\omega\rangle = \frac{1}{2}|\beta\omega\rangle + |\beta\rangle \ , \tag{71}$$

as required. The above OPE also shows that $\beta\omega$ is not a Virasoro primary field, since

$$L_1\beta\omega = -e^{-i\phi}\psi_2 \ . \tag{72}$$

Let us analyze the affine algebraic structure of the module built from the state $|\beta\omega\rangle$. Consider first the action of the current $J^+(z)$ on $\beta\omega(w)$. Extracting the contribution of the first three non-vanishing terms of the OPE yields

$$\begin{aligned} J_2^+\beta\omega &= e^{-3i\phi}\partial^2\psi_2\psi_2 \ , \\ J_1^+\beta\omega &= e^{-3i\phi}[-2i\partial\phi\partial^2\psi_2\partial\psi_2\psi_2 + 3\partial^3\psi_2\partial\psi_2\psi_2] \ , \\ J_0^+\beta\omega &= e^{-3i\phi}\left[\frac{1}{3}\partial^4\psi_2\partial\psi_2\psi_2 - \frac{3}{2}i\partial\phi\partial^3\psi_2\partial\psi_2\psi_2 \right. \\ &\qquad \left. + \ \partial^2\psi_2\partial\psi_2\psi_2[-2(\partial\phi)^2 - i\partial^2\phi]\right] \ . \end{aligned} \tag{73}$$

This computation unravels a part of the indecomposable module $\mathcal{I}_{1/2}$ displayed in Fig. 6. Again, the black dots are associated to the new fields that appear upon extending the model by introducing the symplectic fermions (as opposed to working in the small algebra). The $\beta\omega$ field itself is the black dot labeled $(1/2, 1/2)$. We thus have arrows pointing in the N-E direction, one by two units (corresponding to $J_2^+\beta\omega$ which lies on the extremal diagram), and one by one unit (for $J_1^+\beta\omega$, not drawn), as well as an arrow pointing horizontally to the right ($J_0^+\beta\omega$). The resulting states all contain a symplectic fermion on which no derivative acts. The reached dots are thus all black.

Consider now the action of the current $J^-(z)$ on the three fields just described. At first we have

$$\begin{aligned} J^-(z)J_2^+\beta\omega(w) &= \left(e^{-i\phi}\psi_2\right)(w) + (z-w)\left(e^{-i\phi}[2i\partial\phi\psi_2 - 2\partial\psi_2]\right)(w) \\ &+ \ \mathcal{O}((z-w)^2) \ . \end{aligned} \tag{74}$$

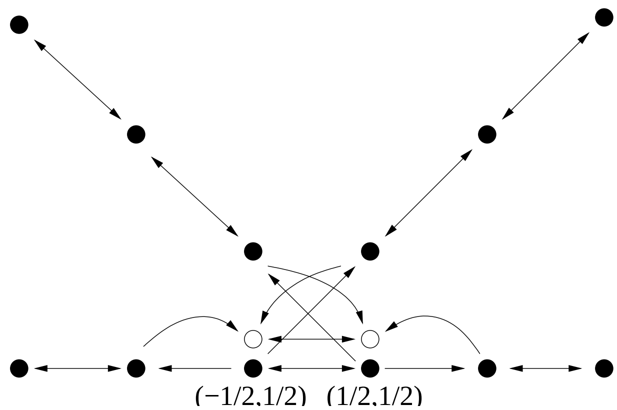

Figure 6: The indecomposable $j = 1/2$ module $\mathcal{I}_{1/2}$.

That $J_0^- J_2^+ \beta\omega = 0$ confirms that the oblique line is really extremal. The J_{-1}^- mode produces the field

$$J_{-1}^- J_2^+ \beta\omega = e^{-i\phi}\psi_2 \ , \tag{75}$$

which has previously been encountered as $L_1\beta\omega$. It is also easily checked to be proportional to $J_1^+\gamma\omega$. Considering the next term in the expansion, we find that $J_{-2}^- J_2^+\beta\omega$ does not return to $\beta\omega$; it is instead expressed as a sum of two fields:

$$J_{-2}^- J_2^+ \beta\omega = 2ie^{-i\phi}\partial\phi\psi_2 - 2e^{-i\phi}\partial\psi_2 \ . \tag{76}$$

The first one is a descendant of $J_{-1}^- J_2^+\beta\omega$. The other one is proportional to β. That confirms the indecomposable character of the representation.

We perform a similar computation on the field $J_1^+\beta\omega$ and find that

$$J_0^- J_1^+ \beta\omega \ \propto \ e^{-i\phi}\psi_2 \ , \tag{77}$$

while $J_{-1}^- J_1^+\beta\omega$ is a linear combination of β and a descendant of $e^{-i\phi}\psi_2$. The corresponding arrows are not drawn since they are not in the extremal part of the diagram. We also find that

$$J_1^- J_0^+ \beta\omega \ \propto \ e^{-i\phi}\psi_2 \ , \tag{78}$$

and again $J_0^- J_0^+\beta\omega$ is a linear combination of β and $J_{-1}^3 e^{-i\phi}\psi_2$. This shows that the arrow pointing from $\beta\omega$ to $J_0^+\beta\omega$ is not two-sided, as indicated. The arrow leaving $J_0^+\beta\omega$ ends on the open dot representing β.

The rest of the extremal diagram is completed straightforwardly. Note that the two new fields $\beta\omega$ and $\gamma\omega$ form a sort of $su(2)$ doublet, albeit immersed into an extended module,

$$J_0^-\beta\omega \ \propto \ \gamma\omega + \cdots \quad , \qquad J_0^+\gamma\omega \ \propto \ \beta\omega + \cdots \ , \tag{79}$$

where the dots refer to a descendant of the state just above $\gamma\omega$ or $\beta\omega$, respectively.

4.5 Closure of the fusion algebra

Twisting the two indecomposable modules just obtained will produce all other indecomposable modules that extend the regular spectrally flowed modules. In particular, we find $\mathcal{I}_{-1/4}$ and $\mathcal{I}_{-3/4}$ by applying π_{-1} to $\mathcal{I}_0$ and $\mathcal{I}_{1/2}$, respectively. This indicates that to every $h = -1/8$ twist field associated to a state at level zero in a highest- or lowest-weight representation, namely τ_n with $n \in \mathbb{Z}$ (see [1], for example)

$$\tau_n = \sigma_n e^{i(n-1/2)\phi} \ , \tag{80}$$

where

$$\sigma_n = \partial\psi_1 \cdots \partial^{n-1}\psi_1 \,, \qquad \sigma_{-n} = \partial\psi_2 \cdots \partial^{n-1}\psi_2 \,, \tag{81}$$

we can associate the Jordan-cell partner $\tau_n\omega$. For instance, $\tau_1 = e^{i\phi/2}$ is Jordan-paired with $\tau_1\omega = e^{i\phi/2}\psi_2\psi_1$. This is easily confirmed by an explicit computation of the OPE of T with $e^{i\phi/2}\psi_2\psi_1$.

At this point in our analysis of the lifted theory, we have found that all admissible representations and their spectral flows have indecomposable Jordan-cell partners. It remains to be understood if the (almost reduced) relaxed modules also have indecomposable Jordan-cell partners. The resolution is obvious, as we have already observed that the relaxed modules correspond to composite fields involving exactly one of the symplectic fermions, ψ_1 or ψ_2, without derivatives. On the other hand, the indecomposable structure is based on Jordan-cell pairs $\Psi\omega$ and Ψ:

$$L_0|\Psi\omega\rangle = h(|\Psi\omega\rangle) + |\Psi\rangle \,, \tag{82}$$

where h denotes the conformal weight. Since a symplectic fermion squares to zero, Ψ cannot contain an underived ψ_i, and hence it cannot correspond to an element in a relaxed module. We may therefore conclude that *there are no Jordan-cell partners to the relaxed modules*. By going from the large algebra to the symplectic formalism, we simply double the number of relaxed modules (see the comment on asymmetry at the end of section 4) and introduce indecomposable representations as Jordan-cell partners to the (ordinary) admissible representations and their twists.

A crucial step toward determining if we now have a complete description of the field content of this $\widehat{su}(2)_{-1/2}$ logarithmic conformal field theory, is to verify that the fusion algebra closes. But closure is an immediate consequence of our free-field construction for the indecomposable representations, and of the closure of the small and large algebra representations.

5 The partition function

Using the free-field representation, we have found that each version of the $\widehat{su}(2)_{-1/2}$ WZW model (based on the small, large or symplectic formulation of the constituent $c = -2$ theory) has a closed fusion algebra. However, this does not automatically ensure that all the fields of the physical theory are accounted for. To address this question, it is traditional in the case of rational CFTs to consider modular invariant partition functions. We try to apply this approach to our problem here, and encounter considerable difficulties.

The basic result underlying most efforts to make sense of WZW models at fractional level is the observation by Kac and Wakimoto [2] that for a given admissible level, there is a finite number of admissible representations, that transform linearly among themselves under modular transformations. This readily leads to a formal modular invariant which is simply the diagonal invariant built out of the admissible *character functions* [9] $F_j(z,q)$ (which are the Kac-Wakimoto admissible characters [2] – whose relationship to the admissible characters χ_j that correspond to genuine power series are reviewed in sect. 2.4). For our case, we are only interested in this diagonal modular invariant[14]:

$$M(z,q) = |F_0(z,q)|^2 + |F_{\frac{1}{2}}(z,q)|^2 + |F_{-\frac{1}{4}}(z,q)|^2 + |F_{-\frac{3}{4}}(z,q)|^2 \,. \tag{83}$$

A natural but complex question is the relation of this expression with the various incarnations of the $\beta\gamma$ system which we have discussed in this and our previous paper [1].

As explained before, the four character functions are mapped onto each other under the spectral flow. Forget for a moment the subtle difference (due to convergence issues) between these character functions and the characters proper. The partition function, defined as usual as the trace over the whole space of states of $q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}z^{J_0^3}\bar{z}^{\bar{J}_0^3}$, for the 'small algebra' $\beta\gamma$ system should then read formally

$$Z = \left(\sum_{\theta=-\infty}^{\infty} 1\right)\left\{|F_0(z,q)|^2 + |F_{1/2}(z,q)|^2 + |F_{-1/4}(z,q)|^2 + |F_{-3/4}(z,q)|^2\right\} \,, \tag{84}$$

[14] The $\widehat{su}(2)_k$ modular invariants have been classified in [12].

where the infinite sum accounts for the sum over all the spectrally flowed representations. As far as modular invariance is concerned, this sum, since it factors out as a numerical prefactor, is irrelevant, and one simply recovers the original Kac and Wakimoto result. Physically, the appearance of the prefactor is a 'reminder' of the divergence of the initial $\beta\gamma$ functional integral, or the unbounded (from below) nature of the spectrum. Actually,, using the character functions instead of the true characters has provided a regularization of this problem.

We now discuss more the issue of characters versus character functions, which is due to 'contact terms'. The key observation is that

$$\frac{1}{1-zq^a} = \begin{cases} \sum_0^\infty z^n q^{na}, & |zq^a|<1 \ , \\ -\sum_1^\infty z^{-n}q^{-na}, & |zq^a|>1 \ . \end{cases} \tag{85}$$

Therefore, inside the circle of convergence, the series coincides with the analytic expression

$$\text{character} = \sum_0^\infty z^n q^{na} = \text{ character function}, \quad |zq^a|<1 \ , \tag{86}$$

while outside, they differ. The character is still defined by the same formal sum, and thus

$$\begin{aligned} \text{character} = \sum_0^\infty z^n q^{na} &= \text{character function} + \sum_{-\infty}^\infty z^n q^{na} \\ &= \text{character function} + \delta(zq^a,1), \quad |zq^a|>1 \ . \end{aligned} \tag{87}$$

The contact terms allow one to express the characters in terms of the character functions. One finds first

$$\begin{aligned} \chi_{j(1,1),\frac{3}{2};0}(z,q) &= F_0(z,q) + q^{1/24}\prod_1^\infty \frac{1}{(1-q^n)^2}\sum_0^N (-1)^p q^{p(p+1)/2}\delta(zq^{2p+1},1) \ , \\ \chi_{j(2,1),\frac{3}{2};0}(z,q) &= F_{1/2}(z,q) + q^{1/24}\prod_1^\infty \frac{1}{(1-q^n)^2}\sum_{-M}^{-1} (-1)^p q^{p(p+1)/2}\delta(zq^{2p+1},1) \ , \end{aligned} \tag{88}$$

where the bounds are as follows. Either z is outside the first disk of convergence, in which case the lower bound is 0 and the higher bound $N \geq 0$ follows from $|q|^{-2N-1} < |z| < |q|^{-2N-3}$; or z is inside the first disk, and then $M \geq 1$ is obtained from $|q|^{2M+1} < |z| < |q|^{2M-1}$.

A similar but different expression holds for the other characters

$$\begin{aligned} \chi_{j(2,2),\frac{3}{2};0}(z,q) &= F_{-1/4}(z,q) + q^{-1/12}\prod_1^\infty \frac{1}{(1-q^n)^2}\sum_1^N (-1)^p q^{p^2/2}\delta(zq^{2p},1) \ , \\ \chi_{j(1,2),\frac{3}{2};0}(z,q) &= F_{-3/4}(z,q) + q^{-1/12}\prod_1^\infty \frac{1}{(1-q^n)^2}\sum_{-M}^{0} (-1)^p q^{p^2/2}\delta(zq^{2p},1) \ . \end{aligned} \tag{89}$$

Here, either z is outside the first disk of convergence, in which case the lower bound is 1 and the higher bound $N \geq 0$ follows from $|q|^{-2N} < |z| < |q|^{-2N-2}$; or z is inside the first disk, and then $M \geq 0$ is obtained from $|q|^{2M+2} < |z| < |q|^{2M}$.

The operator content of the theory includes all the representations obtained by the action of the spectral flow. In some cases, the result is a representation with the spectrum of L_0 bounded from below, but in most cases it is a representation with the spectrum not bounded from below. This spectrum can be deduced from

$$\chi_{j,\frac{3}{2};\theta}(z,q) = q^{-\theta^2/8} z^{-\theta/4} \chi_{j,\frac{3}{2};0}(q, zq^\theta) \ . \tag{90}$$

It so happens however that the contribution of the unbounded part of the spectrum is essentially encoded into contact terms (and as a result, the character functions map nicely onto each other under the action of the flow). For instance, inside the domain $|q| < z < |q|^{-1}$, the character function F_0 coincides with the character of the identity $\chi_{0,\frac{3}{2};0}$. Inside the next domain $|q|^{-1} < |z| < |q|^{-3}$, the character function instead coincides with the spectral flowed object $-\chi_{\frac{1}{2},\frac{3}{2};2}$, since, in the domain of interest, there are no contact terms , that is

$$\chi_{\frac{1}{2},\frac{3}{2};\theta=2} = F_{1/2,\theta=2} = -F_0, \quad |q|^{-1} < |z| < |q|^{-3} \,, \tag{91}$$

and so on.

This suggests another way to define the partition function: one may relax the requirement that 'each operator' is counted for any value of q, z, and instead assume that the operator content is obtained by patching the spectra obtained in each domain of convergence (this idea is also suggested in [6],[9]). For instance, the term $|F_0|^2$ would be interpreted as encoding the untwisted $j = 0$ representation in the first annulus, *plus* the $j = 1/2, w = 2$ twisted representation in the next annulus, etc. At the moment, we have no clear physical justification for this approach, but it seems worth investigating. One then gets the partition function

$$Z(z,q) = \sum_{\substack{j=0,1/2,\\ -1/4,-3/4}} \sum_{\theta \in 2\mathbb{Z}} |\chi_{j,\frac{3}{2};\theta}(z,q)|^2 ch_{j,\frac{3}{2};\theta} \,, \tag{92}$$

where ch is a characteristic function, equal to unity in the annulus of convergence, and 0 otherwise. This almost gives the Kac Wakimoto invariant without the infinite multiplicity prefactor.

The difficulty here arises from the fact that expression (92) provides a covering of the complex z plane (assuming q to be fixed with $|q| < 1$) but only up to the boundaries between the annuli since the region of convergence is given by strict inequalities. One thus realizes that in order to have a full covering of the complex z plane, something is missing: the circles at the boundaries of the different annuli, which are in fact necessary to recover the Kac-Wakimoto invariant.

To get a clearer picture, let us pause to study the effect of modular transformations on the circles of convergence and the poles. Write

$$\tau \to \frac{a\tau + b}{c\tau + d} \,, \qquad \zeta \to \frac{\zeta}{c\tau + d} \qquad\qquad (ad - bc = 1) \,, \tag{93}$$

where $q = e^{2\pi i\tau}$ and $z = e^{2\pi i\zeta}$. Let us start from a point z such that $z = q^n$, $n \in \mathbb{Z}$, which is a pole for half of the (flowed) admissible representations. It means that $\zeta = n\tau + m$ for some integer m. Now consider the modular transformation

$$z \to z' = e^{2\pi i \frac{\zeta}{c\tau+d}} = e^{2\pi i \frac{n\tau+m}{c\tau+d}} \,. \tag{94}$$

To write z' in the form $z' = (q')^N$ for $N \in \mathbb{Z}$ requires

$$\frac{n\tau + m}{c\tau + d} = N \frac{a\tau + b}{c\tau + d} + M \,, \tag{95}$$

for some M. Solving for N and M, using $ad - bc = 1$ yields

$$N = nd - mc \,, \qquad M = ma - nb \,, \tag{96}$$

and these are indeed both integers. We thus fall back on another point that is also a pole for half of the admissible representations. Note, however, that N does not necessarily have the same parity as n. (Recall that when $s = 1$, poles appear for n odd, while for $s = 2$ poles correspond to n even). However, under modular transformation, the fields transform linearly in terms of all the other fields. In particular, the S matrix for this model is

$$S = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & -i & -i \\ 1 & 1 & -i & -i \end{pmatrix} \,, \tag{97}$$

in the order $j = 0, 1/2, -3/4, -1/4$. For instance, F_0 transforms as

$$F_0(-1/\tau, \zeta/\tau) = \frac{1}{2}(F_0(\tau,\zeta) - F_{1/2}(\tau,\zeta) - F_{-3/4}(\tau,\zeta) + F_{-1/4}(\tau,\zeta)) \ . \tag{98}$$

On any circle, half of these representations have poles. That means that regular points can be mapped to poles and vice-versa. So either all the circles in consideration (even the regular ones) are omitted, or all the circles, and thus all the poles, need to be included.

It is a bit uncomfortable mathematically to exclude all the circles from the definition of the partition function. Also, while we had little rationale to begin with to justify the patching of the regions of convergence in the partition function, it seems even more arbitrary to exclude circles where the expressions converge. The only tenable choice seems thus to define the partition function as the Kac-Wakimoto invariant, and try to explain, within this 'patching philosophy' the added circles, including the poles.

The simplest explanation could be that the circles have to be there to make Z well defined mathematically, and do not have a physical meaning. A more complex explanation would be that the circles somehow encode the relaxed representations. While the fact that characters of these representations appear as residues is a tentalizing suggestion in that direction, we have not been able to propose a consistent picture compatible with this interpretation. Were we able to, this would probably show that the $\beta\gamma$ system based on the 'small algebra' is not a consistent theory on the torus, and that the extension to the 'large algebra' is necessary.

A last direction of attack would consist in manipulating formal series. Notice indeed that, using

$$\chi_{0,\frac{3}{2};0} = F_0 + q^{1/24}\prod_1^\infty \frac{1}{(1-q^n)^2}\delta(zq,1), \quad |q|^{-1} < |z| < |q|^{-3} \ , \tag{99}$$

we have

$$\chi_{\frac{1}{2},\frac{3}{2};2} = -\chi_{0,\frac{3}{2};0} + q^{1/24}\prod_1^\infty \frac{1}{(1-q^n)^2}\delta(zq,1) \ , \tag{100}$$

an expression which is now valid irrespective of the values of z, q. The argument generalizes easily. One finds in particular

$$\chi_{\frac{1}{2},\frac{3}{2};-2} = -\chi_{0,\frac{3}{2};0} - q^{1/24}\prod_1^\infty \frac{1}{(1-q^n)^2}\delta(zq^{-1},1) \ , \tag{101}$$

while identical expressions hold by switching the representations $1/2, 0$. One also has

$$\begin{aligned} \chi_{0,\frac{3}{2};1} &= \chi_{-\frac{1}{4},\frac{3}{2};0} \ , \\ \chi_{0,\frac{3}{2};-1} &= -\chi_{-\frac{3}{4},\frac{3}{2};0} + q^{-1/12}\prod_1^\infty \frac{1}{(1-q^n)^2}\delta(z,1) \ , \end{aligned} \tag{102}$$

and identical expressions obtained by switching $1/2, 0$ and similarly $-1/4, -3/4$, etc.

It thus seems plausible that one can write the partition function as a sum of moduli squares of formal series (the characters) instead as the sum of meromorphic functions (the character functions) that we have tried using so far. The implementation of modular invariance and various other questions, however, do not make this approach significantly more fruitful than the others.

To conclude this section: while there are several ways to formally relate the Kac-Wakimoto invariant to our analysis of the $\beta\gamma$ system (including the necessary inclusion of the spectral flow), a detailed analysis of the modular invariant partition function has eluded us. In particular, we have not been able to use the torus to distinguish between the 'small' and 'large' algebra $\beta\gamma$ systems, let alone the 'symplectic algebra' based on the symplectic fermions. Maybe it is not possible.

6 Conclusion

In this work, we have extended our free-field representation analysis of the non-unitary $\widehat{su}(2)_{-1/2}$ WZW model [1]. This representation uses a $c = -2$ $\eta\xi$ ghost system and a Lorentzian boson. In [1] we restricted ourselves to the small algebra. It was found that the spectrum contains an infinite number of fields with arbitrarily large negative dimensions. As a result, the model, in its small algebra formulation, is not a rational CFT, being rather quasi-rational [16] (i.e., there is an infinite but countable number of primary fields, while only a finite number of them appear in a given fusion).

Here we have considered a two-step extension of our previous work: first to the large algebra, and then to the symplectic description of the underlying $c = -2$ theory. At the large algebra level, we have seen that new representations appear. They are identified with some of the relaxed modules of [5, 6]. The relevance of relaxed modules in this context appears to be a new observation.

The symplectic formulation, in turn, involves also indecomposable representations. As demonstrated in some details, they appear in simple OPE calculations based on our free-field representations. The 'emergence' of these indecomposable representations is induced by the introduction of *two* zero modes (those of the symplectic fermions ξ and $\partial^{-1}\eta$). Our observation that the associated indecomposable characters appear to be expressible as the sum of almost reduced (relaxed) modules could potentially be important and deserves a critical study.

At the symplectic level, we find that the overall pattern mimics the one displayed in [3] in the $\widehat{su}(2)_{-4/3}$ case, deduced there by somewhat formal fusion-rule considerations. Our results thus provide independent, albeit implicit, support to the results of [3].

Our analysis has revealed a new example of a logarithmic CFT, the logarithmic lift (i.e., the symplectic version) of the $\widehat{su}(2)_{-1/2}$ WZW model. In turn, the explicit free-field representation offers a concrete way of analyzing it. In fact, we believe that this model could serve as an $c = -1$ counterpart to the paradigmatic $c = -2$ model [17]. The latter has been reviewed in much details in [18] and [19] where also vast lists of references to other relevant works on $c = -2$ may be found.

Let us comment on the relation between these two models – the logarithmic $\widehat{su}(2)_{-1/2}$ model and the $c = -2$ model. It is natural that the logarithmic structure observed in the $\widehat{su}(2)_{-1/2}$ model must be inherited from that of the $c = -2$ model. However, it should be stressed that it is not the 'original' logarithmic structure of the $c = -2$ model that shows up in the $\widehat{su}(2)_{-1/2}$ case. The logarithmic solutions in the $c = -2$ model were found through the analysis of the four-point correlation function of the twist field $\sigma_{1/2}$, of dimension $-1/8$ [17]. This correlator involves two distinct channels (intermediate states) with identical dimensions, leading to an explicit logarithmic solution. That in turn induces a Jordan cell pattern. But this very twist field does not enter in our construction. Indeed, we only use those twist fields of the $c = -2$ model that can be expressed in terms of the $\eta\xi$ fermions. The logarithmic structure (that is, the Jordan cells) arises solely due to the presence of the two zero modes of the symplectic fermions ξ and $\partial^{-1}\eta$, respectively. It would be interesting to understand yet other versions of the associated $\beta\gamma$ system where twist fields in the $\eta\xi$ sector would now be involved.

In relation to the Jordan cells, let us remark that in the present context we only observe the usual Virasoro Jordan cells:

$$L_0|\phi\rangle = h|\phi\rangle + |\psi\rangle\ , \qquad L_0|\psi\rangle = h|\psi\rangle\ , \tag{103}$$

that is, we do not observe any Lie-type Jordan cell of the form

$$J_0^3|\phi\rangle = m|\phi\rangle + |\psi\rangle\ , \qquad J_0^3|\psi\rangle = m|\psi\rangle\ . \tag{104}$$

Whether these can occur in $\widehat{su}(2)$ models remains an interesting open question.

We have also discussed some issues related to the construction of a modular invariant that would match the physical spectrum. This has turned out to be a rather tricky issue, on which no definite conclusions have been presented. In that vein, the symplectic version of the $\widehat{su}(2)_{-1/2}$ model may be the only well-defined one. A further study of the characters and modular invariants may help settling this important question. Indeed, trying to unravel this modular invariant problem in an unambiguous way is a natural extension of this work. Another one is to relate the symplectic free-field representation to the known logarithmic solutions to the KZ equations.

One could also ask to which extend our discussion of the $\widehat{su}(2)_{-1/2}$ model applies to other admissible $\widehat{su}(2)_k$ models. As indicated in section 2.3, the existence of different layer-versions of the theory is quite likely for all cases where $2k$ is an (odd) integer. Recall that, from a fusion-rule point of view, the models with $2k$ integer appear to differ fundamentally from those with $2k$ non-integer. On the other hand, for all (non-integer) admissible k, the top layer is naturally expected to correspond to a logarithmic CFT. This expectation stems from the analysis of two different fractional-level $\widehat{su}(2)_k$ WZW models (with $k = -4/3$ and $k = -1/2$, respectively) based on two quite distinct approaches but with similar results.

Granting the validity of this general expectation, one can argue that these admissible WZW models provide particularly good laboratory models to study logarithmic CFTs. Indeed, they do not embody unfamiliar types of logarithmic structures (i.e., they have Virasoro and not Lie-type Jordan cells) while their skeleton structure, namely the finitely many admissible representations, encodes the crucial modular covariance property.

Although admissible WZW models have yet to be fully tamed, it should be stressed that their use as building blocks of non-unitary CFTs via the diagonal cosets

$$\frac{\widehat{g}_k \oplus \widehat{g}_\ell}{\widehat{g}_{k+\ell}}\ , \qquad (\ell \in \mathbb{Z}_+)\ , \tag{105}$$

nevertheless seems exempt of any ambiguities: complications are found to cancel out nicely between the numerator and the denominator [20].

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A Relaxed representations

A.1 Relaxed modules and charged singular vectors

In this subsection, we introduce the concept of relaxed modules and their twists following [5, 6] (but adopting mainly the notation of the second reference).

Roughly, an untwisted relaxed module is a module defined on a state satisfying the usual highest-weight conditions (the first line of eq. (3) for $\theta = 0$) except for the usual $su(2)$ hightest-weight condition that the state is annihilated by J_0^+, which is not enforced. The term 'relaxed' stems from the fact that one of the defining conditions has been removed or relaxed.

More generally, for a fixed twist parameter, $\theta \in \mathbb{Z}$, the twisted relaxed module $R_{\mu_1,\mu_2,t;\theta}$ is generated by the operators $J^+_{n\le\theta}$, $J^3_{n\le-1}$, and $J^-_{n\le-\theta}$ acting on the vector $|\mu_1,\mu_2,t;\theta\rangle$ satisfying the annihilation conditions

$$J^+_{n\ge1+\theta}|\mu_1,\mu_2,t;\theta\rangle \;=\; J^3_{n\ge1}|\mu_1,\mu_2,t;\theta\rangle \;=\; J^-_{n\ge1-\theta}|\mu_1,\mu_2,t;\theta\rangle \;=\; 0\ , \tag{106}$$

and subject to

$$\begin{aligned} J^-_{-\theta}J^+_\theta|\mu_1,\mu_2,t;\theta\rangle &= -\mu_1\mu_2|\mu_1,\mu_2,t;\theta\rangle\ ,\\ \left(J^3_0+\frac{k}{2}\theta\right)|\mu_1,\mu_2,t;\theta\rangle &= -\frac{1}{2}(1+\mu_1+\mu_2)|\mu_1,\mu_2,t;\theta\rangle\ . \end{aligned} \tag{107}$$

We recall that $t = k + 2$. The Sugawara dimension, $\Delta_{\mu_1,\mu_2,t;\theta}$, and charge, $m_{\mu_1,\mu_2,t;\theta}$, of the twisted relaxed highest-weight vector $|\mu_1,\mu_2,t;\theta\rangle$ are

$$\begin{aligned} \Delta_{\mu_1,\mu_2,t;\theta} &= \frac{(\mu_1-\mu_2)^2-1}{4(k+2)} + \frac{1}{2}(1+\mu_1+\mu_2)\theta + \frac{k}{4}\theta^2\ ,\\ m_{\mu_1,\mu_2,t;\theta} &= -\frac{1}{2}(1+\mu_1+\mu_2) - \frac{k}{2}\theta\ . \end{aligned} \tag{108}$$

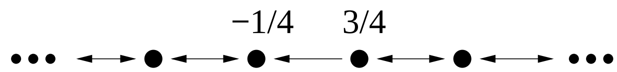

Figure 7: The extremal diagram of the relaxed module $R_{-1,-3/2,3/2}$ with the charged singular vector $J_0^-|-1,-3/2,3/2\rangle$ with charge $m=-1/4$. Note that the left-most part of the diagram is the $su(2)$ highest-weight representation with spin $j=-1/4$.

Note that the construction is symmetric in μ_1 and μ_2. When the twist parameter is omitted, it is understood to be zero and the module or vector is not twisted. Again, comparing the conditions (106) to the similar conditions defining the ordinary (twisted) Verma module (4), we see that one condition has been relaxed in the relaxed module.

A continuous representation (or rather module) of the finite-dimensional Lie algebra $su(2)$ corresponds to the extremal diagram of an untwisted relaxed module with neither μ_1 nor μ_2 integer. It may be useful to recall here the notation that is used in our previous paper [1] as well as in [3]. There continuous representations are denoted by E_s and consist of states $|m\rangle$ with

$$\begin{aligned} J_0^3|m\rangle &= m|m\rangle\ , \\ J_0^+|m\rangle &= |m+1\rangle\ , \\ J_0^-|m\rangle &= \left(C-m(m-1)\right)|m-1\rangle\ , \end{aligned} \tag{109}$$

with

$$C=\frac{(\mu_1-\mu_2)^2-1}{4}\ , \qquad m\in\mathbb{Z}+s\ , \qquad s=-\frac{1}{2}(1+\mu_1+\mu_2)\ . \tag{110}$$

The affine extension is obtained by acting with the negative modes of the currents.

Whenever μ_1 is integer, we have a so-called charged singular vector [5, 6] in the extremal diagram, corresponding to a semi-infinite $su(2)$ representation:

$$\begin{aligned} J_0^+(J_0^-)^{-n}|n,\mu_2,t\rangle &= 0, \qquad\qquad \mu_1=n\in-\mathbb{N}\ , \\ J_0^-(J_0^+)^{n+1}|n,\mu_2,t\rangle &= 0, \qquad\qquad \mu_1=n\in\mathbb{N}_0\ . \end{aligned} \tag{111}$$

The affine extensions are relaxed modules with a charged singular vector in the extremal diagram. An example is illustrated in Fig. 7. Observe that one arrow is not double-sided, signaling the presence of a charged singular vector.

Whenever μ_1 and μ_2 are integers of different signs, the extremal diagram contains two charged singular vectors. Neither can be reached from the other by the action of the zero modes. We shall not be concerned with this case nor with the cases where μ_1 and μ_2 are integers of the same sign, as those cases turn out to be irrelevant for our purposes.

A.2 Singular vectors and embedding diagrams

Singular vectors in the non-extremal diagram have also been examined in [5, 6]. They are essentially obtained by first mapping an extremal vector in the relaxed module to a (possibly twisted) highest-weight vector in an auxiliary Verma module. In the latter module, the singular vectors are known to be (twisted) MFF vectors. When mapped back to the relaxed module, these 'relaxed' MFF vectors satisfy the relaxed highest-weight conditions. The mappings between the relaxed module and the auxiliary Verma module will in general be formal and involve non-integer powers of affine generators. However, the full construction still makes sense [5, 6], in the same way that the original MFF vectors can be shown to correspond to well-defined vectors. For certain values of μ_1 and μ_2 the procedure needs extra caution, but is also treated in [5, 6].

Let us turn to the degeneration patterns of the relaxed modules, $R_{\mu_1,\mu_2,t}$, which will be of interest to us. In the classification provided in [5, 6] (and in the notation of [6]), they are all of type III_+, meaning

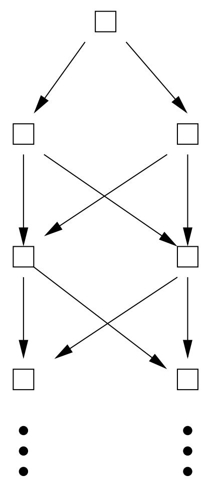

Figure 8: Embedding diagram for a relaxed module of type $\mathrm{III}_+(0)$.

that $t \in \mathbb{Q}_>$, $\mu_1 - \mu_2 \in \mathbb{K}(t)$, and $\mu_1 - \mu_2$, $(\mu_1 - \mu_2)/t \notin \mathbb{Z}$. Here we have borrowed the notation [6]

$$\mathbb{K}(t) = \{a - bt \mid a, b \in \mathbb{Z},\ ab > 0\}\ . \tag{112}$$

There are three interesting sub-cases:
• $\mathrm{III}_+(0)$: $\mu_1, \mu_2 \notin \mathbb{Z}$.
The relaxed module has no charged singular vector. Its embedding diagram is equivalent to the embedding diagram of an ordinary auxiliary Verma module $M_{j,t}$ with spin $j = (\mu_1 - \mu_2 - 1)/2$. This is illustrated in Fig. 8. The corresponding submodules appear at the same levels in R and M, and are nested in identical ways. Relaxed modules of this type are sometimes referred to as continuous representations. It is the only type of relaxed module discussed in [3]. There it is denoted $\mathcal{H}_E$.
• $\mathrm{III}_+(1,-)$: $\mu_1 \in -\mathbb{N}$.
The relaxed module has one charged singular vector. It is the top open dot in Fig. 9, and it generates an ordinary Verma module as a submodule of the relaxed module. Each of the subsequent Verma modules is embedded via a charged singular vector (illustrated by an open dot) into the corresponding relaxed module, having a singular vector (illustrated by an open square) at the same level.
• $\mathrm{III}_+(1,+)$: $\mu_1 \in \mathbb{N}_0$.
The relaxed module has one charged singular vector, and its embedding diagram is obtained by a mirror reflection of the one for type $\mathrm{III}_+(1,-)$. It should be noted that the embedded Verma modules are twisted rather than ordinary Verma modules, with twist parameter $\theta = 1$. Thus, they are simply semi-infinite lowest-weight Verma modules, while the embedded Verma modules in the type $\mathrm{III}_+(1,-)$ diagrams above are semi-infinite highest-weight Verma modules.

A.3 Characters of relaxed representations

The character of the unreduced relaxed module (from which no singular vectors have been factored out) is easy to write [13]:

$$\chi^R_{\mu_1,\mu_2,t;\theta}(z,q) = \delta(zq^{-\theta},1)\frac{q^{\Delta_{\mu_1,\mu_2,t;\theta}}\ z^{m_{\mu_1,\mu_2,t;\theta}}}{\prod_{i\geq 1}(1-q^i)^3}\ , \tag{113}$$

where we have used the notation (43), i.e., $\delta(u,1) = \sum_{n\in\mathbb{Z}} u^n$, and the Sugawara dimension and charge of the twisted relaxed highest-weight state $|\mu_1,\mu_2,t;\theta\rangle$ were introduced in (108). The delta function arises from the boundary of the extremal diagram (which, in an untwisted case, typically corresponds to a continuous representation). Because of it, the character (113) is a formal expression which diverges

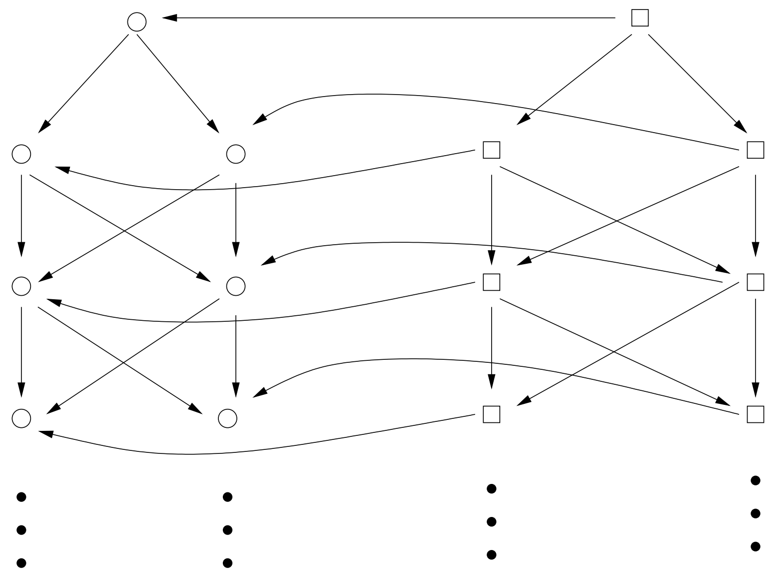

Figure 9: Embedding diagram for a relaxed module of type $\mathrm{III}_+(1,-)$.

for all z. It can be made into a distribution only on circles $|zq^{-\theta}| = 1$ as will be discussed in the last sections. Nevertheless, the delta function notation refers to some convenient features. For instance, one can multiply the expression (113) by integer powers of $zq^{-\theta}$ without changing its (formal) value. The superscript R indicates that we deal with a relaxed module.

Due to the equivalent embedding patterns, characters of irreducible representations $R^\star$ of type $\mathrm{III}_+(0)$ may be obtained in the same way as characters of the auxiliary irreducible representations $M^\star$ (see Fig. 8 and the related discussion). This means that the character of the irreducible relaxed module may be written as a delta function times the residue of a character function associated to an ordinary, irreducible admissible representation. We will give examples of this in the next section.

For characters originating from the type $\mathrm{III}_+(1,-)$ or its mirror-reflected companion, the embedding pattern is also similar to the usual one, as illustrated in Fig. 9. The new feature is the regular submodules generated from the series of charged singular vectors contained in the relaxed (sub-)modules. The character for an irreducible type $\mathrm{III}_+(1,\pm)$ module may thus formally be obtained by subtracting a regular character from a delta function. This in turn may be re-written as the character of a regular representation. This is because the fully reduced module constructed by factoring out all submodules (including the one generated by the charged singular vector in the extremal diagram) is equivalent to a regular module. When the relaxed module is untwisted, the latter regular module corresponds to an ordinary highest- or lowest-weight representation, i.e., when the relaxed module is twisted we have

$$\begin{aligned} \mathrm{III}_+(0,+) \quad &: \quad R^\star_{0,\mu_2,t;\theta} \simeq M^\star_{-(1+\mu_2)/2,t;\theta}\,, \\ \mathrm{III}_+(0,-) \quad &: \quad R^\star_{-1,\mu_2,t;\theta} \simeq M^\star_{(k-\mu_2)/2,t;\theta+1}\,. \end{aligned} \tag{114}$$

Note that we may freely choose the extremal vector with $\mu_1 = 0$ or $\mu_1 = -1$, respectively, to characterize the relaxed module.

A.4 Almost reduced relaxed modules

In the main text, we will consider relaxed modules which are not quite irreducible. They correspond to representations of types $\mathrm{III}_+(1,\pm)$ where the charged singular vectors are *not* removed from the module while all others are. We shall call them *almost reduced* relaxed modules, and denote them by R^{a}. Such a

module is well-defined, and the resulting character will turn out to be a delta function times the residue of a regular representation, i.e., it becomes similar to that of type $\mathrm{III}_+(0)$.

The almost reduced relaxed modules may seem ad-hoc, but they appear naturally in the free-field realization of $\widehat{su}(2)_{-1/2}$, as we discuss below. In order to justify further their presence in the conformal field theory, let us discuss briefly the constraints induced by 'Zhu's algebra'.

A.5 Relaxed modules and Zhu's algebra

It is known that by looking at modes of the first non-trivial null vector of the identity representation, conditions can be obtained on the fields or representations that are allowed in the theory. This consideration of 'equations of motion' is believed to be equivalent to studying the formal Zhu's algebra [14], as described in [15]. We have showed before [1] that there is a null vector, $\mathcal{N}_4$, at level 4 for $k = -1/2$ in the identity module. Expanding the null vector in modes

$$\mathcal{N}_4(z) = \sum_{n\in\mathbb{Z}} V_n(\mathcal{N}_4)\ z^{-n-4}\ , \tag{115}$$

and inserting these modes in three-point functions, we get conditions on representations that are allowed in the theory. For example, applying $V_0(\mathcal{N}_4)$ on a conformal highest-weight state, (i.e., one that is annihilated by $J^a_{n>0}$ in an untwisted module), we get a condition on the allowed untwisted representations of the form

$$(3+16C)[3(J_0^3)^2 - C] = 0\ , \tag{116}$$

where C is the quadratic Casimir

$$C = J_0^- J_0^+ + (J_0^3)^2 + J_0^3\ . \tag{117}$$

Its eigenvalue on the relaxed highest-weight vector is

$$C|\mu_1,\mu_2,t\rangle = \frac{1}{4}\left((\mu_1-\mu_2)^2 - 1\right)|\mu_1,\mu_2,t\rangle\ . \tag{118}$$

The relaxed modules have an infinite number of J_0^3 values and therefore must have $C = -3/16$ to be allowed in the theory. Thus, the only (untwisted) relaxed Verma modules allowed are those having Sugawara or conformal dimension

$$\Delta_{\mu_1,\mu_2,t} = \frac{C}{k+2} = -\frac{1}{8}\ . \tag{119}$$

We also see that the difference $\mu_1 - \mu_2$ is fixed:

$$|\mu_1 - \mu_2| = \frac{1}{2}\ . \tag{120}$$

Similarly, one can derive constraints by acting with other modes, V_n, leading to constraints on the singular vectors in the modules.

Nowhere does the consideration of the 'equations of motion' impose to consider totally reduced relaxed modules. From that perspective, almost reduced (relaxed) modules are therefore allowed.

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